

SPARSE MOMENT SEQUENCES

SAROJ ARYAL, FARHAD JAFARI, AND MIHAI PUTINAR

ABSTRACT. The well-known theorems of Stieltjes, Hamburger and Hausdorff establish conditions on infinite sequences of real numbers to be moment sequences. Further, works by Carathéodory, Schur and Nevanlinna connect moment problems to problems in function theory and functions belonging to various spaces. In many problems associated with realization of a signal or an image, data may be corrupt or missing. Reconstruction of a function from moment sequences with missing terms is an interesting problem leading to advances in image and/or signal reconstruction. It is easy to show that a subsequence of a moment sequence may not be a moment sequence. Conditions are obtained to show how rigid the space of sub-moment sequences is and necessary and sufficient conditions for a sequence to be a sub-moment sequence are established. A deep connection between the sub-moment measures and the moment measures is derived and the determinacy of the moment and sub-moment problems are related. This problem is further related to completion of positive Hankel matrices.

1. INTRODUCTION

While numerous applications of moment problems with a complete set of moments have been identified, they are mostly limited to theoretical observations. For practical implementation of moment problems, it is vital to be able to deal with missing moment data since data obtained from physical experiments and phenomena are often corrupt or incomplete. We call a positive subsequence of a moment sequence a sub-moment sequence and the moment problem of sub-moment sequence a sub-moment problem. In this paper we give a complete characterization of the sub-moment problem and develop a deep connection between the measures arising from the sub-moment problem and the original moment problem. Further, we relate this to completion of positive Hankel matrices and arrive at a necessary and sufficient condition for the completion of such matrices. Recent advances in image reconstruction from sparse MRI data [6], partial data transmission and sparse signals, make this problem timely and of broad interest.

In his memoir "Recherches sur les Fractions Continues" [30] from 1894-95, Stieltjes introduced the "Problem of Moments" on $[0, \infty)$: find a bounded non-decreasing function $\sigma(u)$ in the interval $[0, \infty)$ such that its "moments", s_n , are given as

$$\int_0^\infty u^n d\sigma(u), \quad n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$$

Stieltjes took the terminology "Problem of Moments" from Mechanics. He often used mechanical concepts of mass, density, stability, etc., in solving analytical problems. We can consider $d\sigma(u)$ to be mass distributed over the interval $[u, u + du]$ so that $\int_0^u d\sigma(t)$ gives the mass distributed over $[0, u]$. Accordingly, $\int_0^\infty u d\sigma(u)$ and $\int_0^\infty u^2 d\sigma(u)$ respectively represent the first moment and the second moment, also called moment of inertia,

2010 *Mathematics Subject Classification.* 44A60.

with respect to 0 of the total mass $\int_0^\infty d\sigma(u)$ distributed over the interval $[0, \infty)$. Stieltjes generalizes this concept to call $\int_0^\infty u^n d\sigma(u)$ the n -th moment of the given mass distribution characterized by the function $\sigma(u)$. Stieltjes proves several results concerning solubility of the problem of moments and uniqueness of the solution. He shows that for a sequence $\{s_n\}$ the moment problem is solvable if and only if the determinants of the matrices

$$D_n = \begin{bmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{bmatrix}, D_n^{(1)} = \begin{bmatrix} s_1 & s_2 & \cdots & s_{n+1} \\ s_2 & s_3 & \cdots & s_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1} & s_{n+2} & \cdots & s_{2n+1} \end{bmatrix}, n \in \mathbb{N}_0$$

are non-negative. There have been several sufficiency conditions established for uniqueness of the solution to the moment problem. While Stieltjes had mechanics in mind, Chebyshev and Markov had explicitly probability theory as aims of moment problems.

Hamburger continued the work of Stieltjes in his series of papers "Über eine Erweiterung des Stieltjes Momentenproblems [16]", from 1920-21. He treated the moment problems as a theory of its own. He extended the Stieltjes' moment problem to the whole real axis. The Hamburger Moment Problem is stated as: find a non-decreasing function $\sigma(u)$ in the interval $(-\infty, \infty)$ such that its moments, s_n , are given by

$$\int_{\mathbb{R}} u^n d\sigma(u), n \in \mathbb{N}_0.$$

He established that a necessary and sufficient condition for the moment solution to exist is the positivity of moment sequence. Later, mathematicians including R. Nevanlinna, M. Riesz, T. Carleman, F. Hausdorff, M. Stone, and C. Carathéodory further studied several moment problems.

A moment problem that is not well-posed may not be solvable. Criteria for existence of solution is studied to a great extent for every type of moment problem. Once the solubility criteria are determined, then comes the question of uniqueness of the solution since it is often the case that there is more than one solution. Two solutions of a moment problem are not considered to be distinct if their difference is a constant at all the points of continuity of the difference. Such a moment problem with no more than one distinct solution is called *determinate*, and *indeterminate* otherwise.

Hausdorff's moment problem deals with finding a function $\sigma(u)$ with support on the closed unit interval $[0, 1]$ such that a given sequence $\{s_n\}$ satisfies

$$s_n = \int_0^1 u^n \sigma(u), n \in \mathbb{N}_0.$$

Hausdorff establishes that there is a unique function $\sigma(u)$ in $[0, 1]$ if and only if the sequence $\{s_k\}$ is completely monotonic, that is, it satisfies

$$(-1)^n \Delta^n s_k \geq 0, \text{ for all } n, k \geq 0,$$

where Δ is the difference operator defined as

$$\Delta^n s_k = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} s_{i+k}.$$

This condition for existence and uniqueness holds for the moment problem in any finite interval. In fact, the Hausdorff moment problem is often stated on the interval $[-1, 1]$ just for the sake of simplicity.

The problem of moments stands as a very important problem in analysis up to the present day. Results on moment problems have numerous applications in the areas of extremal problems, optimization theory and limit theorems in probability theory. Rather than the usual infinite moment sequence problem, the truncated moment problem is more applicable in mathematical and physical sciences. Further, multivariable moment problem is more applicable than a single variable one. Multivariate extension of moment problem is also largely studied (see [11] and [26] for instance).

Moments of a geometric object conveys a lot of its topological information such as how round it is, where it is located, what direction it is tapered, where its mass is centered, etc. For simple objects like an ellipsoid, just a few terms in the moment sequence are enough to identify it. However, only infinitely many terms can uniquely identify more complicated shapes like a polygon or a quadrature domain. The authors in [25] give an example of identifying a polygon from its given moments. They prove that the vertices of a nondegenerate, simply connected n -gon in the plane are uniquely determined by its moments up to order $2n - 3$. Theory of moments can also be applied to some modeling problems. An example of speech modeling can be found in [9]. Additional examples of data reconstruction with moments can be found in [10], [14], [15], [19] and [29].

Various generalizations and extensions of the problems of moments have been studied to deal with several applications. Some generalizations include replacing the sequence of functions $\{u^n\}$ with a more general sequence $f_n(u)$ such as $\{e^{inu}\}$, and replacing integrals with more general functionals in abstract spaces. Lasser [23], for example, studies moment problem by replacing $\{u^n\}$ with polynomials. However, we will not consider these generalizations in this paper. The important variation of the moment problem we look at is the moment problem with some of the moments missing. Moment data obtained from various experiments are often corrupt and incomplete. Such reconstruction of moments can be then applied in problems associated with realization of a signal or an image. Reconstruction of a function from moment sequences with missing terms is an interesting problem leading to several advances in image and/or signal reconstruction. In this paper we will work towards finding techniques for reconstructing the missing moment data.

Several authors have studied this problem by replacing the missing moments with an appropriate value, leading to a perturbation of the moment sequences followed by the question of sensitivity of the corresponding orthogonal polynomials. Because of necessary requirements for a sequence to be moments, this approach turns out to be very restrictive in choosing and replacing the missing moments. Thus, we look for ideas to extract a positive subsequence of a positive sequence and look for its moment solution.

We primarily have two questions to answer:

- Q1. How do we characterize positive subsequences of a positive sequence?*
- Q2. How do we characterize the corresponding non-decreasing measures arising from the sub-moment data and what, if any, is the relationship of these measures to the original measures?*

In this paper we establish several methods for extracting positive subsequences and look at solutions to the sub-moment problems. Unless stated otherwise, by *moment problem* we shall refer to the *Hamburger moment problem* throughout the paper. In the first section of this paper, we will thoroughly discuss some classical results and properties of positive sequences and corresponding polynomials. Then we will survey some of the prominent works in the field of moment problem in the next couple of sections. The results described here, mostly without proof and with reference to literature, are those that will play an essential role in the sub-moment problem. The second part of this paper is dedicated to the

study of the sub-moment sequences, sub-moment problems and positive Hankel matrix completion problem. We shall establish necessary and sufficient conditions for a subsequence of a moment sequence to be positive. We will also find conditions for determinacy of sub-moment problems. These results give a complete answer to the problem of Hankel matrix completion.

2. POSITIVE SEQUENCES

Positive sequences play an important role in the theory of orthogonal polynomials, positive functionals and moment problems. This section is concerned with an overview of some characteristics of positive sequences. In this section, we will describe many of the classical theorems and results that we will need in the ensuing sections of this paper. In most cases the proofs are not given and the reader is referred to one of the several paper or classical monographs on the theory. In a few instances, for later reference and for readers' convenience, brief proofs are provided.

Definition 2.1. *An infinite sequence $\{s_k\}$ is positive if the quadratic form*

$$\sum_{i,j=0}^m x_i x_j s_{i+j}$$

is positive for any m and $x_0, x_1, \dots, x_m \in \mathbb{R}$. This is equivalent to saying that the determinant

$$(1) \quad D_m = \begin{vmatrix} s_0 & s_1 & \cdots & s_m \\ s_1 & s_2 & \cdots & s_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_m & s_{m+1} & \cdots & s_{2m} \end{vmatrix}$$

is non-negative for $m \in \mathbb{N}_0$.

Proposition 2.2. *Term-wise sum and product of two positive sequences is positive.*

Proof. The claim that the term-wise sum is positive follows readily from the definition. The claim that the term-wise product is positive follows from the Schur product theorem [18, Theorem 7.5.3]. \square

Given a sequence $\{s_k\}$, define a linear functional Λ such that

$$\Lambda(P(u)) = p_0 s_0 + p_1 s_1 + \cdots + p_n s_n$$

where

$$P(u) = p_0 + p_1 u + \cdots + p_n u^n.$$

Definition 2.3. *A linear functional Λ is called positive if for any polynomial $P : \mathbb{R} \rightarrow \mathbb{R}$, $P(u) \geq 0$ and $P(u) \not\equiv 0$, it follows that $\Lambda(P(u)) > 0$.*

Theorem 2.4. *For any sequence $\{s_k\}$, the functional Λ defined above is positive if and only if $\{s_k\}$ is positive.*

Proof. Suppose $\{s_k\}$ is positive. Any polynomial $P(u) = p_0 + p_1 u + \cdots + p_n u^n \geq 0$ and $P(u) \not\equiv 0$ can be written in the form

$$P(u) = [A(u)]^2 + [B(u)]^2$$

where $A(u), B(u)$ are polynomials with real coefficients. Then we have

$$\Lambda(P(u)) = \sum_{i,j=0}^m x_i x_j s_{i+j} + \sum_{i,j=0}^m y_i y_j s_{i+j} \geq 0.$$

Now suppose Λ is positive. Then

$$\sum_{i,j=0}^m x_i x_j s_{i+j} = \Lambda([P(u)]^2),$$

where $P(u) = x_0 + x_1 u + \cdots + x_m u^m$. Hence $\{s_k\}$ is positive. \square

Given a positive sequence $\{s_k\}$, we can construct a set of polynomials $\{P_n(u)\}$ such that $P_n(u)$ is of degree n with positive leading coefficient and they are orthonormal with respect to the functional Λ . These polynomials are explicitly given by the formula

$$(2) \quad P_n(u) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \cdots & s_{n-1} & s_n \\ s_1 & s_2 & \cdots & s_n & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} & s_{2n} \\ 1 & u & \cdots & u^{n-1} & u^n \end{vmatrix},$$

where

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \text{ and } D_{-1} = 1.$$

It is well known that this orthonormal set of polynomials $\{P_n(u)\}$ satisfy the three term recurrence relation

$$(3) \quad uP_n(u) = b_n P_{n+1}(u) + a_n P_n(u) + b_{n-1} P_{n-1}(u), \quad n = 1, 2, 3, \dots,$$

where

$$a_n = \Lambda(u[P_n(u)]^2), \text{ and } b_n = \frac{\sqrt{D_{k-1}D_{k+1}}}{D_k},$$

and

$$P_0(u) = 1, \text{ and } P_1(u) = \frac{u - a_0}{b_0}.$$

Together with the orthonormal polynomials $\{P_n(u)\}$, the polynomials of the second kind $\{Q_n(u)\}$ are also generated from the same three-term recurrence relation (3) with initial conditions $Q_0(u) = 0, Q_1(u) = \frac{1}{b_0}$. $\{P_n(u)\}$ and $\{Q_n(u)\}$ are linearly independent solutions to (3). In fact, $Q_n(u)$ is a polynomial of degree $n - 1$ and can be expressed explicitly as

$$(4) \quad Q_n(u) = \Lambda\left(\frac{P_n(u) - P_n(v)}{u - v}\right).$$

Orthonormal sets of polynomials $\{P_n(u)\}$ and $\{Q_n(u)\}$ play an important role in the theory of moment problems. See [13] for further details on the polynomials and their recurrence relation. Several conditions on these polynomials have been established to determine the existence and uniqueness of solutions of moment problems. We will elaborate more on the concepts and usage of these polynomials in the later sections.

3. CLASSICAL MOMENT PROBLEMS & DETERMINACY

In this section we review some of the key results in the Hamburger's moment problem, also called power moment problem, which is stated as: given an infinite sequence of numbers $\{s_k\}$, find a non-decreasing function $\sigma(u)$ such that

$$(5) \quad s_k = \int_{\mathbb{R}} u^k \sigma(u).$$

This problem was formulated and solved by Hamburger. Later works by mathematicians including Fischer, Akheizer and Krein establish conditions on the spectrum of the moment solutions. Some of these results are summarized in the theorems below. The reader is referred [1] and [28] for an extensive account of the principle results.

Theorem 3.1 (Hamburger). *There is a non-decreasing function $\sigma(u)$ with an infinite number of points of increase such that equation (5) holds if and only if $\{s_k\}$ is positive.*

Theorem 3.2. *For a non-decreasing function $\sigma(u)$ with precisely $k+1$ distinct points of increase such that equation (5) holds, it is necessary and sufficient that*

$$D_0, D_1, \dots, D_k > 0, D_{k+1} = D_{k+2} = \dots = 0,$$

where D_i is the determinant of the Hankel matrix defined in (1).

For a proof of the Hamburger Theorem, refer to [1]. The proof uses the Helly's Theorem extensively. In some instances, such as in [24] and [31], the properties of the determinants D_m have been studied to describe the moment solution.

The orthonormal set of polynomials $\{P_n(u)\}$ and $\{Q_n(u)\}$ are used to establish the conditions on determinacy of the moment problems. So before moving on to results on determinacy we will define the circle $K_n(\lambda)$ as follows.

Theorem 3.3 (Hellinger). *Let $\lambda \in \mathbb{C}$ be fixed with $\Im \lambda > 0$ ($\Im \lambda < 0$) and let $\tau \in \mathbb{R}$. Then*

$$(6) \quad w_n(\lambda, \tau) = -\frac{Q_n(\lambda) - \tau Q_{n-1}(\lambda)}{P_n(\lambda) - \tau P_{n-1}(\lambda)} = -\frac{Q_n(\lambda, \tau)}{P_n(\lambda, \tau)}$$

describes a circular contour $K_n(\lambda)$ in the open half plane $\Im w > 0$ ($\Im w < 0$). The center of this circle is at the point

$$(7) \quad -\frac{Q_n(\lambda)\overline{P_{n-1}(\lambda)} - Q_{n-1}(\lambda)\overline{P_n(\lambda)}}{P_n(\lambda)\overline{P_{n-1}(\lambda)} - P_{n-1}(\lambda)\overline{P_n(\lambda)}}$$

with radius

$$(8) \quad \frac{1}{|\lambda - \bar{\lambda}| \sum_{k=0}^{n-1} |P_k(\lambda)|^2}.$$

Furthermore, the equation of the circle can be written in the form

$$(9) \quad \frac{w - \bar{w}}{\lambda - \bar{\lambda}} - \sum_{k=0}^{n-1} |w P_k(\lambda) + Q_k(\lambda)|^2 = 0$$

Proof. Refer to [17]. □

Note that $K_{n+1}(\lambda) \subseteq K_n(\lambda)$. We call the limiting circle $K_\infty(\lambda)$. It can be shown that the quantity

$$(10) \quad w(\lambda) = \int_{\mathbb{R}} \frac{d\sigma(u)}{u - \lambda}$$

falls on the circumference of $K_\infty(\lambda)$, [1, p. 30-34].

Now we look at some results concerning determinacy of the Hamburger moment problem. Most of the determinacy theorems we talk about here deal with the nature of the limiting circle $K_\infty(\lambda)$. It is useful to present the following theorem and its proof on determinacy since we will be using it for the case of sub-moment problem in a later section.

Theorem 3.4. *If $\{s_k\}$ is a positive sequence such that its corresponding limiting circles $K_\infty(\lambda)$ is a point, then the Hamburger moment problem of sequence $\{s_k\}$ is determinate.*

Proof. Suppose the moment problem has two solutions $\sigma_1(u)$ and $\sigma_2(u)$. Consider the functions

$$(11) \quad f_1(u) = \int_{\mathbb{R}} \frac{d\sigma_1(u)}{u-\lambda}, \text{ and } f_2(u) = \int_{\mathbb{R}} \frac{d\sigma_2(u)}{u-\lambda},$$

for some $\lambda \in \mathbb{C}$ such that $\Im \lambda \neq 0$. Now for a fixed λ , each of $f_1(\lambda)$ and $f_2(\lambda)$ are in $K_\infty(\lambda)$ by Theorem 2.2.4 in [1]. But if $K_\infty(\lambda)$ is a point, then $f_1(\lambda)$ and $f_2(\lambda)$ coincide. Putting

$$\alpha(u) = \sigma_1(u) - \sigma_2(u),$$

we have

$$\int_{\mathbb{R}} \frac{d\alpha(u)}{u-\lambda} = 0.$$

By the Stieltjes and Perron inversion formula,

$$\frac{\alpha(u-0) + \alpha(u+0)}{2} = c \quad \text{for some constant } c.$$

Therefore $\sigma_1(u)$ and $\sigma_2(u)$ represent the same solution of the moment problem. \square

The converse of this theorem is also true. That is, if the limiting circle $K_\infty(\lambda)$ is not a point, then the Hamburger moment problem is indeterminate. Carleman[7] proved several striking results regarding determinacy of the Hamburger moment problem. We mention a couple of them here.

Theorem 3.5 (Carleman). *If for a positive sequence $\{s_k\}$,*

$$\sum_{n=1}^{\infty} \frac{1}{b_n} = \infty,$$

where b_n are constants as given in the three term recurrence relation (3), then the moment problem is determinate.

Theorem 3.6 (Carleman). *If for a positive sequence $\{s_k\}$,*

$$\sum_{n=1}^{\infty} s_{2n}^{-\frac{1}{2n}} = \infty,$$

then the moment problem is determinate.

4. FUNCTION THEORY AND THE MOMENT PROBLEMS

For a measure $\sigma(u)$ which arises as a moment solution to a positive sequence, we can define the L^p space in the usual sense:

$$(12) \quad L^p_\sigma = \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : f \text{ is } \sigma\text{-measurable and } \int_{\mathbb{R}} |f(u)|^p d\sigma(u) < \infty \right\}.$$

The space of functions L_σ^p with the norm

$$(13) \quad \|f(u)\|_{L_\sigma^p} = \left(\int_{\mathbb{R}} |f(u)|^p d\sigma(u) \right)^{\frac{1}{p}}$$

is complete and is a Hilbert Space when $p = 2$ where inner product defined as

$$(14) \quad (f(u), g(u))_\sigma = \int_{\mathbb{R}} f(u) \overline{g(u)} d\sigma(u)$$

for any functions $f, g \in L_\sigma^p$.

Use of the space L_σ^2 to study the properties such as determinacy of the solution(s) $\sigma(u)$ for the moment problem is very prevalent. This is done mostly in connection to the polynomials generated by the corresponding sequence $\{s_k\}$. Several theorems by M. Riesz characterize solutions of a moment problem and its determinacy [27]. Below is a very useful consequence of these theorems.

Corollary 4.1. *If σ is the solution of a determinate moment problem, then the set of all the polynomials is dense in L_σ^2 .*

The Nevanlinna formula relates all the solutions of an indeterminate moment problem to the functions in N -class. This formula establishes a one-to-one correspondence between the aggregate of all the solutions σ of an indeterminate moment problem and the aggregate of all the functions ϕ in class N , which we will define below, augmented by the constant at ∞ . Applying the Stieltjes-Perron inversion formula to the Nevanlinna formula one can determine σ in terms of ϕ .

Definition 4.2. *Nevanlinna matrix is any matrix of the form*

$$\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}$$

where the elements $a(z), b(z), c(z)$ are entire transcendental functions, and the following two conditions are satisfied:

- (1) $a(z)d(z) - b(z)c(z) = 1$
- (2) For any fixed real number $\tau \in \mathbb{R}$ the function

$$w(z) = \frac{\tau a(z) - c(z)}{\tau b(z) - d(z)}$$

is regular in each open half plane and satisfies

$$\frac{\Im w(z)}{\Im z} > 0.$$

Now, let us consider the four families of polynomials given as

$$\begin{aligned} A_n(z) &= z \sum_{k=0}^{n-1} Q_k(0) Q_k(z) \\ B_n(z) &= -1 + z \sum_{k=0}^{n-1} Q_k(0) P_k(z) \\ C_n(z) &= 1 + z \sum_{k=0}^{n-1} P_k(0) Q_k(z) \\ D_n(z) &= z \sum_{k=0}^{n-1} P_k(0) P_k(z). \end{aligned}$$

If the moment problem is indeterminate, then the sums

$$\sum_{k=0}^{\infty} |P_k(z)|^2, \sum_{k=0}^{\infty} |Q_k(z)|^2$$

are bounded and converge uniformly to respective limits on compact subsets of \mathbb{C} . By the Cauchy-Bunyakovskii inequality, the polynomials $A_n(z), B_n(z), C_n(z), D_n(z)$ converge uniformly to the entire transcendental functions

$$\begin{aligned} A(z) &= z \sum_{k=0}^{\infty} Q_k(0) Q_k(z) \\ B(z) &= -1 + z \sum_{k=0}^{\infty} Q_k(0) P_k(z) \\ C(z) &= 1 + z \sum_{k=0}^{\infty} P_k(0) Q_k(z) \\ D(z) &= z \sum_{k=0}^{\infty} P_k(0) P_k(z) \end{aligned}$$

on compact subsets of \mathbb{C} .

It has already been shown that the limit functions $A(z), B(z), C(z), D(z)$ are entire transcendental. It can also be shown that the functions $A(z), B(z), C(z), D(z)$ satisfy the requirements for elements of a Nevanlinna matrix. Thus the matrix

$$\begin{bmatrix} A(z) & B(z) \\ D(z) & D(z) \end{bmatrix}$$

is a Nevanlinna matrix.

Before we state the Nevanlinna formula, we define the Nevanlinna class, N-class, of functions.

Definition 4.3 (N-class). *The Nevanlinna class (Class N) consists of all holomorphic functions $w = f(z)$ on the upper half plane such that $\Im w > 0$. By the Riesz-Herglotz integral representation formula, a function $f \in N$ may be expressed as*

$$f(z) = \mu z + \nu + \int \frac{1+uz}{u-z} d\tau(u)$$

where $\mu, \nu \in \mathbb{R}, \mu \geq 0$ and $\tau(u)$ is a non-decreasing function with bounded variation.

Theorem 4.4. *If $\Im z > 0$, for an indeterminate moment problem,*

$$\int \frac{d\sigma(u)}{u-z} = -\frac{A(z)\phi(z)-C(z)}{B(z)\phi(z)-D(z)},$$

where $A(z), B(z), C(z), D(z)$ form a Nevanlinna matrix, σ is a solution to the concerned moment problem, and $\phi \in N$.

This theorem leads to the question of relation between the moment solutions of subsequences to the functions in the N-class. It is worthwhile to look for the moment solutions of subsequences in the same N-class.

5. INCOMPLETE AND SPARSE MOMENT SEQUENCE

There are two main methods for dealing with missing moments: (i) Perturbation/modification, and (ii) Subsequences. Perturbation or modification method is done by replacing the missing data with an appropriate value, whereas the method of subsequences is based on simply removing the missing moments. In some applications, people have tried replacing the lost moments with zero as well. Because of Hamburger's theorem, in both cases the modified data has to be positive to guarantee a moment solution. Let us look at an example of a positive sequence and some of its perturbations and subsequences.

The sequence given by $s_k = \frac{1}{k+1}$ is a positive sequence since, for every k , determinant of Hankel matrix is simply given as

$$D_k = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \cdots & \frac{1}{2k+1} \end{vmatrix} = \prod_{i=0}^k \left[(2i+1) \binom{2i}{i}^2 \right],$$

which is positive. So the sequence $\{s_k\}$ is positive and has a moment solution.

By setting $s_3 = 0$, the second Hankel matrix has determinant

$$D_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} < 0.$$

This shows how easily we lose positivity of the sequence if perturbations of modifications is not done properly. Thus, perturbation is a very restrictive method because of the requirement of positivity. Due to this restriction and involvement of extensive matrix computations, the study of sensitivity of corresponding polynomials and moment solutions becomes computationally expensive, if not intractable. In this chapter we will mostly deal with the method of subsequences.

The subsequence given by $s_{k_i} = \frac{1}{k_i!} \in \{s_k\}$ is not a positive sequence since the determinant of its second Hankel matrix is

$$D_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} \end{vmatrix} < 0$$

and hence has no solution to its moment problem. This example shows that an arbitrary subsequence of a positive sequence is not necessarily positive. Thus we need to look for ideas to appropriately remove moments while preserving positivity of subsequences.

5.1. Perturbation of a Positive Sequence. In this section we will make a few notes regarding perturbations of terms in a positive sequence. Modification of a term in a moment sequence very significantly affects its moment solution and orthogonal polynomial it generates. Several people have studied the sensitivity of orthogonal polynomials to modifications of the moments. Gautschi [12], for example, studies sensitivity of orthogonal polynomials to perturbations in moments. Reader is advised to refer to [4] and [22] for some more examples on modified moments.

The fact that $s_0 > |s_i|$, $i = 1, 2, 3, \dots$, suggests that we can have a wide range of choices for s_0 keeping the rest of terms constant and maintaining positivity. However, we are not much interested in modifying s_0 since we can always scale this term. The theorem below shows how rigid positive sequences are with respect to a single term s_n , $n \neq 0$.

Theorem 5.1. *Let $\{s_k\}_{k \geq 0}$ be a positive sequence. If*

$$\tilde{s}_k \begin{cases} s_k & \text{if } k \neq n \\ \tau & \text{if } k = n \end{cases}$$

is positive and $\tilde{s}_k \leq s_k$, then $\tau = s_n$.

Proof. Consider an arbitrary system of number $x_0, x_1, x_2, \dots, x_m$. Then we can write the sum

$$(15) \quad \sum_{i,j=0}^m x_i x_j s_{i+j} = \sum_{\substack{i,j=0 \\ i+j=n}}^m x_i x_j s_n + \sum_{\substack{i,j=0 \\ i+j \neq n}}^m x_i x_j s_{i+j} \geq 0.$$

Similarly, the positivity of $\{\tilde{s}_k\}_{k \geq 0}$ gives

$$(16) \quad \sum_{i,j=0}^m x_i x_j \tilde{s}_{i+j} = \sum_{\substack{i,j=0 \\ i+j=n}}^m x_i x_j \tau + \sum_{\substack{i,j=0 \\ i+j \neq n}}^m x_i x_j s_{i+j} \geq 0.$$

Combining (15) and (16), and using $\tilde{s}_k \leq s_k$,

$$\sum_{\substack{i,j=0 \\ i+j=n}}^m x_i x_j \tau - \sum_{\substack{i,j=0 \\ i+j=n}}^m x_i x_j s_n + \sum_{i,j=0}^m x_i x_j s_{i+j} \geq 0,$$

which implies

$$(17) \quad \sum_{i,j=0}^m x_i x_j s_{i+j} \geq [s_n - \tau] \sum_{\substack{i,j=0 \\ i+j=n}}^m x_i x_j.$$

Since $\{s_k\}_{k \geq 0}$ is a positive sequence the left side of (17) should be positive for any system of numbers $x_0, x_1, x_2, \dots, x_m$. Now, let $s_n > \tau$ and choose $x_0, x_1, x_2, \dots, x_m = 1$. Then

$$\sum_{\substack{i,j=0 \\ i+j=n}}^m x_i x_j = (n+1). \text{ Then (17) gives } \sum_{i,j=0}^m x_i x_j s_{i+j} \geq (n+1)(s_n - \tau) > 0. \text{ But this is not true}$$

since, scaling the sequence $\{s_k\}$ to have the first term as $s_0 = \frac{(n+1)(s_n - \tau)}{2(m+1)^2}$ and using the fact that $s_i \leq s_0$ for all i , we have

$$\sum_{i,j=0}^m x_i x_j s_{i+j} \leq \sum_{i,j=0}^m x_i x_j s_0 = (m+1)^2 s_0 = \frac{(n+1)(s_n - \tau)}{2} < (n+1)(s_n - \tau).$$

Therefore $s_n \leq \tau$.

Similarly using (15) and (16) again we can obtain

$$(18) \quad \sum_{\substack{i,j=0 \\ i+j=n}}^m x_i x_j s_{i+j} \geq [\tau - s_n] \sum_{\substack{i,j=0 \\ i+j=n}}^m x_i x_j.$$

Then using the same argument as above we see that $s_n \geq \tau$. This completes the proof. \square

It is noteworthy that if we perturb more than one term then we have more flexibility. The moment sequence tends to get less rigid as we replace more terms. If the moments s_1 and s_2 are replaced by x_1 and x_2 respectively, does imposing positivity imply $s_1 = x_1, s_2 = x_2$? Suppose $s_{p_i}, i = 1, 2, \dots, n$ are replaced by τ_{p_i} . Then we can write the sum

$$(19) \quad \sum_{i,j=0}^m x_i x_j s_{i+j} = \sum_{q=1}^n \sum_{\substack{i,j=0 \\ i+j=p_q}}^m x_i x_j s_{p_q} + \sum_{\substack{i,j=0 \\ i+j \neq p_1, \dots, p_n}}^m x_i x_j s_{i+j} \geq 0.$$

Similarly, the positivity of $\{\tilde{s}_k\}_{k \geq 0}$ gives

$$(20) \quad \sum_{i,j=0}^m x_i x_j \tilde{s}_{i+j} = \sum_{q=1}^n \sum_{\substack{i,j=0 \\ i+j=p_q}}^m x_i x_j \tau_{p_q} + \sum_{\substack{i,j=0 \\ i+j \neq p_1, \dots, p_n}}^m x_i x_j s_{i+j} \geq 0.$$

Combining (19) and (20), and using $\tilde{s}_k \leq s_k$ gives

$$(21) \quad \sum_{q=1}^n \left[\xi_{p_q} \sum_{\substack{i,j=0 \\ i+j=p_q}}^m x_i x_j \right] \leq \sum_{i,j=0}^m x_i x_j s_{i+j}$$

where $\xi_{p_q} = s_{p_q} - \tau_{p_q}$. Now setting $x_i = 1$ for $i = 0, 1, \dots, m$ from (21) we have

$$(22) \quad \sum_{q=1}^n [\xi_{p_q} (p_q + 1)] \leq (m+1)^2 s_0.$$

Although very restrictive and not much revealing, perturbation can be useful in some cases. An example of an application of modified moments to harmonic solids is given in [5]. An important application of modification in s_0 is that it can lead to a determinate solution. It was proved in Stieltjes' 1894-Memoir [30] that a determinate moment solution can be obtained from an indeterminate one with a modification in s_0

The moment sequence $s_n = q^{\frac{-(n+1)^2}{2}}$ gives the Stieltjes-Wigert polynomials $P_n(u; q)$, which are orthogonal in a log-normal distribution, known to be indeterminate. The modified sequence $\{\tilde{s}_k\}_{k \geq 0}$ defined as

$$\tilde{s}_0 = s_0 - \frac{1}{\sum_{n=0}^{\infty} [P_n(0; q)]^2}$$

and $\tilde{s}_n = s_n$ for all $n \geq 1$ has a determinate moment solution $\tilde{\sigma}(u)$ given by

$$\tilde{\sigma}(u) = \sum_{u \in U} c_u \delta_u,$$

where U is the zero set of the reproducing kernel

$$K(0, w) = \sum_{n=0}^{\infty} P_n(0) P_n(w)$$

and

$$c_u = \frac{1}{\sum_{k=0}^{\infty} [P_k(u; q)]^2}, u \in U.$$

5.2. Positive Subsequences. As was seen in an above example, an arbitrary subsequence of a positive sequence is not necessarily positive. We will develop some methods to extract positive subsequences from a positive sequence.

First we give a simple construction of a positive sequence by the following theorem and proposition, which can then be used to generate positive subsequences.

Theorem 5.2. *Let $f_k : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of functions such that*

$$(23) \quad f_i(u) f_j(u) = f_{i+j}(u).$$

Then the sequence $\{s_k\}$ given as

$$s_k = \int f_k(u) d\sigma(u)$$

is positive for any non-decreasing function $\sigma(u)$ with an infinite number of points of increase.

Proof. For any finite set of numbers x_0, x_1, \dots, x_m , we have

$$\begin{aligned} \sum_{i,j=0}^m x_i x_j s_{i+j} &= \sum_{i,j=0}^m x_i x_j \int f_{i+j}(u) d\sigma(u) \\ &= \sum_{i,j=0}^m x_i x_j \int f_i(u) f_j(u) d\sigma(u) \\ &= \int \sum_{i,j=0}^m x_i x_j f_i(u) f_j(u) d\sigma(u) \\ &= \int \left[\sum_{i=0}^m x_i f_i(u) \right]^2 d\sigma(u) \geq 0. \end{aligned}$$

Note that the equality sign holds if the function $\sum_{i=0}^m x_i f_i(u)$ is zero at every point where $\sigma(u)$ increases. Since $\sigma(u)$ increases at infinitely many points then the function $\sum_{i=0}^m x_i f_i(u)$ is identically zero. Therefore we have the equality sign only if $x_i = 0$ for all i . \square

Proposition 5.3. *The sequence $f_k(u)$ satisfies the condition $f_i(u)f_j(u) = f_{i+j}(u)$ for $i, j \in \mathbb{N}_0$ if and only if*

$$f_k(u) = (f_1(u))^k.$$

Proof. Clearly the conclusion holds for $k = 1$. Assume $f_n(u) = (f_1(u))^n$ holds. Then $f_{n+1}(u) = f_n(u)f_1(u) = (f_1(u))^n f_1(u) = (f_1(u))^{n+1}$, inductively. Proof of the converse is trivial. \square

Note that Theorem 5.2 and Proposition 5.3 allow many constructions of positive sequences. For example, letting $f_k(u) = u^k$ we obtain precisely the classical power moment sequence for any non-decreasing measure $\sigma(u)$. Similarly, letting $f_k(u)$ to be a^{ku} for some constant a or $(\phi(u))^k$ for any σ -measurable function ϕ gives us more positive sequences corresponding to a non-decreasing measure $\sigma(u)$.

Now we are interested in extracting positive subsequences from a moment sequence. Using Theorem 5.2 and Proposition 5.3, we can construct several positive subsequences from a moment sequence. For example, let $\{s_k\}$ be a moment sequence constructed from a non-decreasing measure $\sigma(u)$ defined as

$$s_k = \int (\phi(u))^k d\sigma(u)$$

for some function $\phi(u)$. Then for a fixed $\ell \in \mathbb{N}_0$ the sequence

$$s_{k_\ell} := \int ((\phi(u))^\ell)^k d\sigma(u) = \int (\phi(u))^{k_\ell} d\sigma(u) = s_{k_\ell}$$

is a positive subsequence of $\{s_k\}$. But how can we characterize all the positive subsequences we can extract from a moment sequence? Given a positive sequence $\{s_k\}$, the problem of identifying all the positive subsequences requires finding all the sequences $\{\ell_k\} \subseteq \mathbb{N}_0$ such that the sequence given as

$$\tilde{s}_k = s_{k+\ell_k}$$

is positive.

Theorem 5.4. *Let $\{s_k\}$ be a positive sequence. Then the sequence $\tilde{s}_k = s_{k+\ell_k}$ is a positive sequence if $\{u^{\ell_k}\}$ is a positive sequence for all $u \in \mathbb{R}$.*

Proof. By Hamburger's theorem $\{s_k\}$ is a positive sequence is equivalent to the fact that there is a non-decreasing function $\sigma(u)$ such that

$$s_k = \int_{\mathbb{R}} u^k d\sigma(u), \quad k \in \mathbb{N}_0.$$

For $x_0, x_1, \dots, x_m \in \mathbb{R}$ we have

$$\begin{aligned}
 \sum_{i,j=0}^m x_i x_j \tilde{s}_{i+j} &= \sum_{i,j=0}^m x_i x_j s_{i+j+\ell_{i+j}} \\
 &= \sum_{i,j=0}^m x_i x_j \int_{\mathbb{R}} u^{i+j+\ell_{i+j}} d\sigma(u) \\
 &= \int_{\mathbb{R}} \left[\sum_{i,j=0}^m x_i x_j u^{i+j+\ell_{i+j}} \right] d\sigma(u) \\
 &= \int_{\mathbb{R}} \left[\sum_{i,j=0}^m u^{\ell_{i+j}} (x_i u^i) (x_j u^j) \right] d\sigma(u).
 \end{aligned}
 \tag{24}$$

But the integrand in (24) can be written as

$$\sum_{i,j=0}^m u^{\ell_{i+j}} (x_i u^i) (x_j u^j) = \begin{bmatrix} x_0 & x_1 u & \cdots & x_m u^m \end{bmatrix} \begin{bmatrix} u^{\ell_0} & u^{\ell_1} & \cdots & u^{\ell_m} \\ u^{\ell_1} & u^{\ell_2} & \cdots & u^{\ell_{m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ u^{\ell_m} & u^{\ell_{m+1}} & \cdots & u^{\ell_{2m}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 u \\ \vdots \\ x_m u^m \end{bmatrix}.
 \tag{25}$$

Thus, the integral (24) is positive if the Hankel matrix for the sequence $\{u^{\ell_k}\}$ is positive for all $u \in \mathbb{R}$. This is equivalent to the sequence $\{u^{\ell_k}\}$ being positive for all $u \in \mathbb{R}$. \square

However, the following example due to David Kimsey (private correspondence) shows that the converse of this theorem does not hold in general. We gratefully acknowledge this contribution.

Consider the sequence $\{s_k\}$ where $s_k = \frac{1}{2^k}$. It is easy to verify that this sequence is positive. Notice that $\{\tilde{s}_k\}$, where

$$\tilde{s}_k = s_{k+1} = \frac{1}{2^{k+1}} = \left(\frac{1}{2}\right) \frac{1}{2^k},$$

is a positive sequence, but $\{u^{\ell_k}\}$, where $\ell_k = 1$, $k \in \mathbb{N}_0$, is not a positive sequence in general for $u \in \mathbb{R}$. Indeed, if we put $u = -1$ then we arrive at a negative sequence.

It is easy to see that the difficulty pointed out by this example is that u can take negative values. Restricting u to nonnegative values in \mathbb{R} (i.e. the Stieltjes moment problem) would have made the converse of the above theorem to also hold. Therefore, it should not be surprising that for even powers of u stronger results hold. Now we give the following corollaries which give further insights on the characteristics of the sequence $\{\ell_k\}$ for extracting a positive subsequence.

Corollary 5.5. *Let $\{s_k\}$ be a positive sequence. Then the sequence $\{\tilde{s}_k\}$ defined by $\tilde{s}_k = s_{k+\ell}$ for any $\ell \in 2\mathbb{N}_0$ is a positive sequence.*

Proof. Note that the constant sequence $\{u^\ell\}$ for $\ell \in 2\mathbb{N}_0$ is positive for any $u \in \mathbb{R}$. Apply Theorem 5.4. \square

Corollary 5.6. *If the sequence $\{u^{\ell_k}\}$ is positive then*

- (1) $\ell_i \in 2\mathbb{N}_0$ for every $i \in 2\mathbb{N}_0$, and
- (2) for every odd positive integer i , $\ell_i = \frac{\ell_{i-1} + \ell_{i+1}}{2}$.

Proof. By Corollary 5.5, since every tail of the sequence $\{u^{\ell_k}\}$ with ℓ_k starting from an even integer is positive,

$$u^{\ell_i} \geq 0 \text{ for every } u \in \mathbb{R}.$$

Hence $\ell_i \in 2\mathbb{N}_0$ for every $i \in 2\mathbb{N}_0$.

Again using Corollary 5.5, for every $(i-1) \in 2\mathbb{N}_0$, the two by two matrix of the form

$$\begin{bmatrix} u^{\ell_{i-1}} & u^{\ell_i} \\ u^{\ell_i} & u^{\ell_{i+1}} \end{bmatrix}$$

must have nonnegative determinant. Then

$$(26) \quad u^{\ell_{i-1}+\ell_{i+1}} - u^{2\ell_i} \geq 0.$$

Since the inequality (26) is true for every $u \in \mathbb{R}$, the statement (2) follows. \square

The converse of this corollary does not hold in general. To see this, consider the sequence $\{u^{\ell_k}\}$ with $\{\ell_k\} = \{0, 1, 2, 4, 6, \dots\}$. The determinant of the corresponding 3×3 Hankel is $-u^6(u-1)^2$, which is a negative function.

Theorem 5.7. *Let $\{s_k\}$ be a positive sequence. The subsequence $\{\tilde{s}_k\}$ given as $\tilde{s}_k = s_{k+\ell_0}$ is positive if $\ell_k = kd + \ell_0$ for any $d \in \mathbb{N}_0$, and $\ell_0 \in 2\mathbb{N}_0$.*

Proof. Suppose $\ell_k = kd + \ell_0$ where $d \in \mathbb{N}_0$, and $\ell_0 \in 2\mathbb{N}_0$. Then for all real numbers $x_0, x_1, x_2, \dots, x_m$, we have

$$\begin{aligned} \sum_{i,j=0}^m x_i x_j \tilde{s}_{i+j} &= \sum_{i,j=0}^m x_i x_j s_{i+j+(i+j)d+\ell_0} \\ &= \sum_{i,j=0}^m x_i x_j \int_{\mathbb{R}} u^{(1+d)i+(1+d)j+\ell_0} d\sigma(u) \\ &= \int_{\mathbb{R}} \left[u^{\frac{\ell_0}{2}} \sum_{i=0}^m x_i u^{(1+d)i} \right]^2 d\sigma(u) \geq 0. \end{aligned}$$

\square

The above results show that any subsequence of a positive sequence is positive if they are extracted in a certain periodic manner. A converse to these results will be proved in a later section using matrix completion method. Specifically, we will establish that extraction of a subsequence from a positive sequence must be in a periodic manner for the subsequence to be positive. Here we provide an example. Consider the sequence given by

$$s_k = \frac{1}{(k+1)e^{k+1}}.$$

This sequence is positive since, for every m , the determinant of the Hankel matrix given by

$$D_m = \begin{vmatrix} \frac{1}{e} & \frac{1}{2e^2} & \cdots & \frac{1}{(m+1)e^{m+1}} \\ \frac{1}{2e^2} & \frac{1}{3e^3} & \cdots & \frac{1}{(m+2)e^{m+2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m+1)e^{m+1}} & \frac{1}{(m+2)e^{m+2}} & \cdots & \frac{1}{(2m+1)e^{2m+1}} \end{vmatrix} = \frac{1}{e^{(m+1)^2}} \prod_{i=0}^m \left[(2i+1) \binom{2i}{i}^2 \right],$$

is positive. But the subsequence given as

$$\tilde{s}_k = \frac{1}{(k+1)^2 e^{(k+1)^2}}$$

is not positive since the terms are not taken in a periodic manner as defined by the above theorem. We can easily verify this by looking at the determinant of the second Hankel matrix:

$$D_2 = \begin{vmatrix} \frac{1}{e} & \frac{1}{2e^2} \\ \frac{1}{2e^2} & \frac{1}{4e^4} \end{vmatrix} < 0.$$

6. SUB-MOMENT PROBLEMS

Given any two positive sequences $\{s_k\}$ and $\{\tilde{s}_k\}$ with moment solutions $\sigma(u)$ and $\tilde{\sigma}(u)$ respectively, $\sigma(u) = \tilde{\sigma}(u)$ if and only if $s_k = \tilde{s}_k$ for all k . Therefore the moment solutions to any two distinct positive sequences can never be equal. If $\{\tilde{s}_k\}$ is a positive subsequence of $\{s_k\}$ with corresponding moment solutions $\tilde{\sigma}(u)$ and $\sigma(u)$ respectively, then these solutions can not be the same. In this section, we develop techniques to compare $\sigma(u)$ and $\tilde{\sigma}(u)$.

Consider a positive sequence $\{s_k\}$ with its moment solution $\sigma(u)$. If some of the terms from $\{s_k\}$ are missing then we obtain a subsequence which, if positive, gives another moment solution. For an appropriately chosen $\{\ell_k\} \subseteq \mathbb{N}_0$, assume that $\{\tilde{s}_k\} = \{s_{k+\ell_k}\}$ is positive and has the moment solution $\tilde{\sigma}(u)$. Write

$$(27) \quad \int_{\mathbb{R}} u^k d\tilde{\sigma}(u) = \int_{\mathbb{R}} u^{k+\ell_k} d\sigma(u).$$

Then for any polynomial $P(u) = \sum_{k=0}^n a_k u^k$, we have

$$(28) \quad \int_{\mathbb{R}} P(u) d\tilde{\sigma}(u) = \int_{\mathbb{R}} P_{\ell_k}(u) d\sigma(u),$$

where $P_{\ell_k}(u) = \sum_{k=0}^n a_k u^{k+\ell_k}$.

Here, we present some results concerning moment solution to a shifted sequence $\tilde{s}_k = s_{k+\ell}$, $\ell \in 2\mathbb{N}_0$, which has already been proved to be a positive subsequence of a positive sequence s_k .

Theorem 6.1. *Let $\{s_k\}$ be a positive sequence and $\{\tilde{s}_k\}$ be its subsequence given as $\tilde{s}_k = s_{k+\ell}$ for a fixed $\ell \in 2\mathbb{N}_0$. Let $\sigma(u)$ and $\tilde{\sigma}(u)$ respectively be the moment solutions of the sequences $\{s_k\}$ and $\{\tilde{s}_k\}$. If one of the moment problems is determinate, then*

$$(29) \quad \int_{\mathbb{R}} f(u) u^\ell d\sigma(u) = \int_{\mathbb{R}} f(u) d\tilde{\sigma}(u), \quad f \in L_\sigma^2 \cap L_{\tilde{\sigma}}^2.$$

Proof. To prove the first assertion, define the following two linear functionals:

$$\Phi_1(f(u)) = \int_{\mathbb{R}} f(u) u^\ell d\sigma(u), \quad \Phi_2(g(u)) = \int_{\mathbb{R}} g(u) d\tilde{\sigma}(u)$$

for $f \in L_\sigma^2$ and $g \in L_{\tilde{\sigma}}^2$. To see Φ_1 and Φ_2 are bounded, we have

$$\Phi_1(f(u)) \leq \int_{\mathbb{R}} |f(u) u^\ell| d\sigma(u) \leq \|f(u)\|_{L_\sigma^2} \sqrt{s_{2\ell}} < \infty$$

and

$$\Phi_2(f(u)) \leq \int_{\mathbb{R}} |g(u)| d\tilde{\sigma}(u) \leq \|g(u)\|_{L_{\tilde{\sigma}}^2} \sqrt{s_0} < \infty.$$

Now for any function $f \in L_{\sigma}^2 \cap L_{\tilde{\sigma}}^2$,

$$\Phi(f(u)) = \Phi_1(f(u)) - \Phi_2(f(u))$$

is a bounded linear functional.

Observe that for any polynomial $P(u) = a_m u^m + \dots + a_1 u + a_0$,

$$\begin{aligned} \Phi(P(u)) &= \int_{\mathbb{R}} P(u) u^{\ell} d\sigma(u) - \int_{\mathbb{R}} P(u) d\tilde{\sigma}(u) \\ &= \int_{\mathbb{R}} [a_m u^{m+\ell} + \dots + a_1 u^{1+\ell} + a_0 u^{\ell}] d\sigma(u) - \int_{\mathbb{R}} [a_m u^m + \dots + a_1 u + a_0] d\tilde{\sigma}(u) \\ &= a_m (s_{m+\ell} - \tilde{s}_m) + \dots + a_1 (s_{1+\ell} - \tilde{s}_1) + a_0 (s_{\ell} - \tilde{s}_0) \\ &= 0. \end{aligned}$$

By Corollary 4.1 and due to the determinacy condition of the moment problems, the set of polynomials is dense in L_{σ}^2 or $L_{\tilde{\sigma}}^2$. Hence the set of polynomials is dense in $L_{\sigma}^2 \cap L_{\tilde{\sigma}}^2$. Therefore, by the Hahn-Banach Theorem, Φ is identically zero. Thus equation (29) holds. \square

The above theorem is particularly useful when the first finitely many moments are missing. It says that we can find the moment solution with the remaining data and reconstruct the missing moments by using formula (29). Whether the moment solution of a subsequence is determinate or not is to be studied more carefully in comparison with the determinacy of the moment solution of the original sequence. It is, however, known that the first term in the subsequence $\{s_{k\ell}\}$ can be modified to make the moment problem determinate, but the solution will not remain the same.

As a remark to L^2 space of sub-moment solutions, the following theorem gives us a relation between L^2 spaces related to a moment sequence and its sub-moment sequence. Here we require the evenness assumption on ℓ to guarantee that ν is a nondecreasing measure.

Corollary 6.2. *Consider a positive sequence $\{s_k\}$ and its subsequence $\{\tilde{s}_k\} = \{s_{k+\ell}\}$ for $\ell \in 2\mathbb{N}_0$. Let $\sigma(u)$ and $\tilde{\sigma}(u)$ be the moment solutions of the sequence and the subsequence respectively. Then there is a measure $\nu(u)$ absolutely continuous with respect to $\sigma(u)$ such that*

$$(30) \quad (L_{\sigma}^2 \cap L_{\tilde{\sigma}}^2) \subseteq L_{\nu}^2.$$

Proof. Let $f \in L_{\sigma}^2 \cap L_{\tilde{\sigma}}^2$. By equation (29) we have

$$\int_{\mathbb{R}} f(u) u^{\ell} d\sigma(u) = \int_{\mathbb{R}} f(u) d\tilde{\sigma}(u), \quad f \in L_{\sigma}^2 \cap L_{\tilde{\sigma}}^2.$$

Set $d\nu = u^{\ell} d\sigma$. Then $\nu \ll \sigma$ and

$$(31) \quad \int_{\mathbb{R}} f(u) d\nu(u) = \int_{\mathbb{R}} f(u) d\tilde{\sigma}(u).$$

Hence $f \in L_{\nu}^2$. \square

Now consider a general sub-moment sequence $\{\tilde{s}_k\}$ given as $\tilde{s}_k = s_{k+\ell_k}$ for an appropriate $\{\ell_k\} \subseteq \mathbb{N}_0$. It was shown in the previous section that for any k ,

$$\ell_k = kd + \ell_0, \text{ where } d \in \mathbb{N}_0 \text{ and } \ell_0, \ell_1 \in 2\mathbb{N}_0.$$

Modifying the functional Φ_1 in the proof of Theorem 6.1 gives the following result.

Theorem 6.3. *Let $\{s_k\}$ be a positive sequence and $\{\tilde{s}_k\}$ be its subsequence given by $\tilde{s}_k = s_{kd+\ell_0}$. Let $\sigma(u)$ and $\tilde{\sigma}(u)$ respectively be the moment solutions of the sequences $\{s_k\}$ and $\{\tilde{s}_k\}$. If one of the moment problems is determinate, and $f(u^d) \in L^2_\sigma$ for any $f \in L^2_\sigma$, then*

$$(32) \quad \int_{\mathbb{R}} f(u^d) u^{\ell_0} d\sigma(u) = \int_{\mathbb{R}} f(u) d\tilde{\sigma}(u), \quad f \in L^2_\sigma \cap L^2_{\tilde{\sigma}}.$$

A precise relation between L^2 spaces of moment solutions to the original sequence and that of one of its positive subsequences can be useful in characterizing the sub-moment solutions and hence approximating and/or reconstructing the missing data.

Exploring equation (32) a little further, for $\lambda \in \mathbb{C}, \Im \lambda \neq 0$, define

$$f(u) = \frac{1}{u - \lambda}.$$

Then we have

$$\int_{\mathbb{R}} |f(u)|^2 d\sigma(u) = \int_{\mathbb{R}} \frac{d\sigma(u)}{|u - \lambda|^2} \leq \int_{\mathbb{R}} \frac{d\sigma(u)}{y^2} = \frac{s_0}{y^2} < \infty.$$

since $|u - \lambda| \geq |y|$. Therefore, $f \in L^2_\sigma$. By the same argument, $f \in L^2_{\tilde{\sigma}}$. Similarly, it can be shown that $f(u^d) \in L^2_\sigma$. Then by Theorem 6.3,

$$(33) \quad \int_{\mathbb{R}} \frac{u^{\ell_0} d\sigma(u)}{u^d - \lambda} = \int_{\mathbb{R}} \frac{d\tilde{\sigma}(u)}{u - \lambda}.$$

Applying Theorem 4.4 to the equation (33) yields the following result which connects a moment problem with the polynomials corresponding to its sub-moment problem.

Theorem 6.4. *Let $\{s_k\}$ be a positive sequence and $\{\tilde{s}_k\}$ be its subsequence given as $\tilde{s}_k = s_{kd+\ell_0}$. Let $\sigma(u)$ and $\tilde{\sigma}(u)$ respectively be the moment solutions of the sequences $\{s_k\}$ and $\{\tilde{s}_k\}$. If the moment problem of $\{\tilde{s}_k\}$ is indeterminate, then*

$$(34) \quad \int_{\mathbb{R}} \frac{u^{\ell_0} d\sigma(u)}{u^d - \lambda} = -\frac{\tilde{A}(\lambda)\phi(\lambda) - \tilde{C}(\lambda)}{\tilde{B}(\lambda)\phi(\lambda) - \tilde{D}(\lambda)},$$

where $\tilde{A}(\lambda), \tilde{B}(\lambda), \tilde{C}(\lambda), \tilde{D}(\lambda)$ form a Nevanlinna matrix of the sub-moment problem, and $\phi \in N$.

Next, for the shifted sub-sequence we discussed in Theorem 6.1, we want to investigate determinacy of its moment problem of a special sub-moment sequence in relation to determinacy of the original moment problem. Recall that the limit circles $K_\infty(\lambda), \lambda \in \mathbb{C}, \Im \lambda \neq 0$ defined in the Section 2 provide a fundamental concept for studying determinacy of moment problems.

Theorem 6.5. *For the same moment sequences and conditions as in Theorem 6.1, for $\lambda \in \mathbb{C}, \Im \lambda \neq 0$ the following holds.*

$$(35) \quad \int_{\mathbb{R}} \frac{d\tilde{\sigma}(u)}{u - \lambda} = C + \lambda^\ell \int_{\mathbb{R}} \frac{d\sigma(u)}{u - \lambda}.$$

where C is a constant depending on λ .

Proof. It is already shown that, for

$$f(u) = \frac{1}{u - \lambda}, \quad \lambda \in \mathbb{C}, \Im \lambda \neq 0,$$

$f \in L^2_\sigma \cap L^2_{\tilde{\sigma}}$. Then by Theorem 6.1,

$$\int_{\mathbb{R}} \frac{u^\ell d\sigma(u)}{u-\lambda} = \int_{\mathbb{R}} \frac{d\tilde{\sigma}(u)}{u-\lambda}.$$

But $\frac{u^\ell}{u-\lambda} = R(u) + \frac{\lambda^\ell}{u-\lambda}$, where $R(u) = \sum_{i=0}^{\ell-1} \lambda^i u^{\ell-1-i}$. Therefore,

$$\int_{\mathbb{R}} \frac{d\tilde{\sigma}(u)}{u-\lambda} = \int_{\mathbb{R}} R(u) d\tilde{\sigma}(u) + \lambda^\ell \int_{\mathbb{R}} \frac{d\sigma(u)}{u-\lambda}.$$

Hence

$$\int_{\mathbb{R}} \frac{d\tilde{\sigma}(u)}{u-\lambda} = C + \lambda^\ell \int_{\mathbb{R}} \frac{d\sigma(u)}{u-\lambda},$$

where $C = \sum_{i=0}^{\ell-1} \lambda^i s_{\ell-1-i}$. □

Let

$$w_\sigma(\lambda) = \int_{\mathbb{R}} \frac{d\sigma(u)}{u-\lambda} \text{ and } w_{\tilde{\sigma}}(\lambda) = \int_{\mathbb{R}} \frac{d\tilde{\sigma}(u)}{u-\lambda}.$$

Then Theorem 6.5 states that

$$(36) \quad w_{\tilde{\sigma}}(\lambda) = C + \lambda^\ell w_\sigma(\lambda).$$

for s_k and \tilde{s}_k specified in that theorem.

Recall that points w_σ and $w_{\tilde{\sigma}}$ lie on the circumferences of the circles $K_\infty(\lambda)$ and $\tilde{K}_\infty(\lambda)$ corresponding to the moment problems of $\{s_k\}$ and $\{\tilde{s}_k\}$ respectively. Equation (36) precisely describes the distortion of the circle corresponding to the original moment sequence in relation to its tail.

7. COMPLETION OF POSITIVE HANKEL MATRIX

Recall from Definition 2.1 that the positivity of an infinite sequence $\{s_k\}$ is equivalent to the positivity of Hankel matrices of the form

$$H_m = \begin{bmatrix} s_0 & s_1 & \cdots & s_m \\ s_1 & s_2 & \cdots & s_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_m & s_{m+1} & \cdots & s_{2m} \end{bmatrix}.$$

Determining whether a subsequence of a positive sequence $\{s_k\}$ is positive is then equivalent to a matrix completion problem. A partial Hankel matrix is a partial matrix that is Hankel to the extent to which it is specified. All the specified entries lie along certain skew-diagonals (positive sloping diagonals), and all the entries along a specified skew-diagonal have the same value. For example, for a moment sequence $\{s_k\}$ with every $(2i+1)$ -th entry missing we have a corresponding Hankel matrix

$$\tilde{H}_5 = \begin{bmatrix} s_0 & * & s_2 & * & s_4 & * \\ * & s_2 & * & s_4 & * & s_6 \\ s_2 & * & s_4 & * & s_6 & * \\ * & s_4 & * & s_6 & * & s_8 \\ s_4 & * & s_6 & * & s_8 & * \\ * & s_6 & * & s_8 & * & s_{10} \end{bmatrix}.$$

Essentially, we want to determine whether a particular partial Hankel matrix can be completed to a positive Hankel matrix. Note that here we are interested in completing the partial Hankel matrices corresponding to a sparse moment sequence.

The problem of matrix completion has been studied in great details (see [2], [3], [6], [20], for example). Most common matrices that are of interest in matrix completion are positive definite matrices, positive semi-definite matrices, co-positive matrices, totally positive matrices and Toeplitz matrices. Matrix completion properties have several applications in sparse moment problem. Some results on multidimensional trigonometric moment problem have been proved using moment completion in [2].

Most of the results in matrix completion deal with the case when the main diagonal entries are specified. This creates a problem in completing a partial Hankel matrix. Whenever a $(2i)$ -th term of a moment sequence is missing, the corresponding Hankel matrix has a diagonal entry missing.

Definition 7.1. *Given an $n \times n$ matrix A , let w, v be subsets of \mathbb{N}_0 . A submatrix $A[w, v]$ consists of the entries in rows w and columns v . The principle submatrix is $A[w] = A[w, w]$.*

For example,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \quad A[\{2, 4\}] = A[\{2, 4\}, \{2, 4\}] = \begin{bmatrix} 6 & 8 \\ 14 & 16 \end{bmatrix}.$$

Matrix completion encounters an important inheritance structure. If A is a matrix with property X , then in most cases of matrix completion of interest, the fully specified principle submatrices of A inherit the property X . Positive (semi) definite matrix completions have this inheritance structure. But the converse is not true for a general positive (semi) definite matrix. That is, a partial matrix with all of its fully specified principle submatrices positive (semi) definite does not necessarily have a positive (semi) definite completion, [20]. Similarly, if a partial matrix \tilde{A} admits a completion of rank $\leq k$, then every specified submatrix of A has rank $\leq k$.

With the inheritance structure of matrix completion, the matrix completion problem for positive Hankel matrices becomes equivalent to finding all the positive subsequences of a moment sequence. That is, the principle sub-Hankel matrices of a positive Hankel matrix are those corresponding to positive subsequences of a moment sequence.

Given a Hankel matrix H corresponding to a moment sequence $\{s_k\}$ consider a fully specified principle submatrix $H[\alpha]$, where $\alpha = \{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \mathbb{N}_0$. For $H[\alpha]$ to be also a Hankel matrix we require that

$$\alpha_{i_1} + \alpha_{j_1} = \alpha_{i_2} + \alpha_{j_2} \text{ whenever } i_1 + i_2 = j_1 + j_2.$$

Then for any k , we have

$$\alpha_k = \frac{\alpha_{k-1} + \alpha_{k+1}}{2}.$$

Also note that every principal submatrix $H[\alpha]$ has $(1, 1)$ entry s_{2i} for some i . With this observation and Theorem 5.7, we have proved the following result:

Theorem 7.2. *Let $\{s_k\}$ be a positive sequence. A subsequence $\{\tilde{s}_k\}$ given as $\tilde{s}_k = s_{k+\ell_k}$, is positive if and only if $\ell_k = kd + \ell_0$ for some $d \in \mathbb{N}_0$ and $\ell_0 \in 2\mathbb{N}_0$.*

In terms of matrix completion the above theorem gives a necessary and sufficient condition for completion of a partial Hankel matrix. Interestingly, this result for Hankel matrices

seems to be similar with the result for Toeplitz matrices proved in [21]. The reader is referred to [3] for more on matrix completion problem for Hankel and Toeplitz matrices.

8. REMARKS

The problem of identifying positive subsequences of moment sequences and the corresponding matrix completion problem is the forward problem associated with sparse sequences and sparse matrices. The corresponding inverse problem of extension of positive sequences as sparse subsets of moment sequences is closely related. The necessary and sufficient conditions obtained suggests that positive sparse sequences can be completed by extending the measures associated with these sequences by Theorem 6.4 and calculating the arising moments. This suggests that the moment sequence can be recovered to a great degree from its positive sparse subsequences. The entropy concept introduced by J. Chover [8] for extensions of positive definite functions was used to find a unique extension of positive definite functions which maximizes the entropy. In parallel to this classical and well-known theory, we anticipate that for each positive sequence arising from the Nevanlinna class there is a unique extension in the Nevanlinna class that maximizes the entropy. The corresponding phrasing of this problem for positive Hankel matrices gives a norm, induced by the entropy, which allows recovery of these structured matrices from their sparse positive submatrices. Relationship of this approach to finding the matrix with minimum nuclear norm that fits the data in Candès and Recht's [6] is not clear to us at the present time, but is a subject of further investigation.

REFERENCES

1. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Translated by N. Kemmer, Hafner Publishing Co., New York, 1965.
2. Mihály Bakonyi and Geir Naevdal, *On the matrix completion method for multidimensional moment problems*, Acta Sci. Math. (Szeged) **64** (1998), no. 3-4, 547–558.
3. Mihály Bakonyi and Hugo J. Woerdman, *Matrix Completions, Moments, and Sums of Hermitian Squares*, Princeton University Press, Princeton, NJ, 2011.
4. Bernhard Beckermann and Emmanuel Bourreau, *How to choose modified moments?*, J. Comput. Appl. Math. **98** (1998), no. 1, 81–98.
5. Carl Blumstein and John C. Wheeler, *Modified-moments method: Applications to harmonic solids*, Phys. Rev. B. **8** (1973), no. 4, 1764–1776.
6. Emmanuel Candès and Benjamin Recht, *Exact matrix completion via convex optimization*, Found. Comput. Math. **9** (2009), no. 6, 717–772.
7. P.L. Carleman, *Les Fonctions Quasi-Analytiques. Collection Borel*, Gauthier-Villars, Paris, 1926.
8. J. Chover, *On normalized entropy and the extensions of a positive-definite function*, J. Math. Mech. **10** (1961), 927–945.
9. Ph. Delsarte, Y. Genin, Y. Kamp, and P. Van Dooren, *Speech modelling and the trigonometric moment problem*, Philips J. Res. **37** (1982), no. 5-6, 277–292.
10. Ky Fan, *On positive definite sequences*, Ann. of Math. (2) **47** (1946), 593–607.
11. Bent Fuglede, *Multidimensional moment problem*, Exposition. Math. **1** (1983), no. 1, 47–65.
12. Walter Gautschi, *On the sensitivity of orthogonal polynomials to perturbations in the moments*, Numer. Math. **48** (1986), no. 4, 369–382.
13. C. D. Godsil, *Algebraic combinatorics*, Chapman and Hall Mathematics Series, Chapman & Hall, New York, 1993.
14. Rao S. Govindaraju and Bhabani S. Das, *Moment Analysis for Subsurface Hydrologic Applications*, Springer, Dordrecht, The Netherlands, 2007.
15. Björn Gustafsson, Mihai Putinar, Edward B. Saff, and Nikos Stylianopoulos, *Bergman polynomials on an archipelago: estimates, zeros and shape reconstruction*, Adv. Math. **222** (2009), no. 4, 1405–1460.
16. Hans Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Ann. **82** (1920, 1921), 120–164, 168–187.
17. Ernst Hellinger, *Zur Stieltjesschen Kettenbruchtheorie*, Math. Ann. **86** (1922), no. 1-2, 18–29.

18. Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.
19. Aleš Jaklič and Franc Solina, *Moments of superellipsoids and their application to range image registration*, IEEE Transactions on Systems, Man, and Cybernetics **33** (2003), no. 4, 648–657.
20. Charles R. Johnson, *Matrix completion problems: a survey*, Matrix theory and applications (Phoenix, AZ, 1989), Proc. Sympos. Appl. Math., vol. 40, Amer. Math. Soc., Providence, RI, 1990, pp. 171–198.
21. Charles R. Johnson, Michael Lundquist, and Geir Nævdal, *Positive definite Toeplitz completions*, J. London Math. Soc. (2) **59** (1999), no. 2, 507–520.
22. Sun-Mi Kim and Lothar Reichel, *Sensitivity analysis for Szegő polynomials*, Numer. Math. **113** (2009), no. 2, 265–279.
23. Rupert Lasser, *On the problem of modified moments*, Proc. Amer. Math. Soc. **90** (1984), no. 3, 360–362.
24. Bruce G. Lindsay, *On the determinants of moment matrices*, Ann. Statist. **17** (1989), no. 2, 711–721.
25. Peyman Milanfar, Mihai Putinar, James Varah, Bjorn Gustafsson, and Gene Golub, *Shape reconstruction from moments: Theory, algorithms, and applications*, Proceedings of SPIE Conference 4116, Advanced Signal Processing Algorithms, Architectures, and Implementations **4116** (2000), 406–416.
26. Mihai Putinar and Florian-Horia Vasilescu, *Solving moment problems by dimensional extension*, Ann. of Math. (2) **149** (1999), no. 3, 1087–1107.
27. M. Riesz, *Sur le problème des moments et le théorème de parseval correspondant*, Acta Litt. ac., Azeged **1** (1922), 209–225.
28. J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, American Mathematical Society Mathematical surveys, vol. II, American Mathematical Society, New York, 1943.
29. Barry Simon, *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math. **137** (1998), no. 1, 82–203.
30. Thomas Jan Stieltjes, *Œuvres complètes/Collected papers. Vol. I, II*, Springer-Verlag, Berlin, 1993, Reprint of the 1914–1918 edition.
31. Harold Widom, *Hankel matrices*, Trans. Amer. Math. Soc. **121** (1966), 1–35.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WYOMING, LARAMIE, WY 82070

E-mail address: saryal@uwyo.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WYOMING, LARAMIE, WY 82070

E-mail address: fjafari@uwyo.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT SANTA BARBARA, SANTA BARBARA, CA 93106

E-mail address: mputinar@math.ucsb.edu